Attribution

Modeling asset characteristics as portfolios.

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I present a flexible, unified, and portfolio-centered approach to attribution. It treats attribution questions that occur before the fact (ex ante) and after the fact (ex post) in an entirely symmetric manner. Ex ante we are concerned with the sources of risk in the portfolio, the sources of possible outperformance, and whatever may be hindering us from a more efficient implementation. Ex post we are concerned with the performance of our forecasts, the performance of the portfolio, and the efficacy of our risk control.

I look at all these questions in terms of an analysis of the covariance (or correlation) of the return between two portfolios. This requires an unusual step to model asset characteristics such as alphas and returns as portfolios. Then I model the covariance of those characteristic portfolios and the portfolios that we actually hold.

The benefit of making this unfamiliar step of modeling asset characteristics as portfolios is that it provides us an ability to look at all questions in a similar way. We can achieve a great economy of thought by concentrating in the portfolio domain.

There are two main elements in my approach:

- Framing interesting portfolio management questions in terms of an analysis of covariance.
- Analyzing the covariance.

I start by showing how to model several aspects of a portfolio, risk, alpha, the transfer coefficient, and expected loss in utility as proportional to a covariance.

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Then I present a rudimentary analysis of risk predictions when there is little structure available. I add structure by detailing a set of possible sources that are likely to be of help in explaining the composition of our portfolio, and show how the same analysis used to attribute alpha and risk ex ante can be used ex post to attribute return. Finally, I demonstrate the flexibility of the idea by showing how other return-regression schemes fall into this framework.

There is a vast amount of information available on attribution. A Google search on *portfolio attribution* gets 172,000 hits. My research is based on the cross-sectional return-regression framework of Fama and MacBeth [1973], Rosenberg and McKibben [1973], and Fama and French [1993].

I also use the insight of Clarke, de Silva, and Thorley [2002, 2005] that implementation efficiency is neatly captured by the predicted return correlation between the portfolio you hold and the portfolio you would like to hold, absent cost and constraints. Clarke, de Silva, and Thorley link the ex ante with the ex post, and use the realized information coefficient as the ex post analogue of the transfer coefficient.

My notion of the backlog is similar in spirit to their "optimal active weight not taken," and my concept of the opportunity set is akin to their realized dispersion measure. The motivation and the approach, however, are different. I am trying to provide a framework—think of laboratory test equipment—that facilitates study of a wide range of topics on risk, implementation efficiency, and return. Indeed, my initial motivation was the alpha vintage model (described later). In pursuing that limited objective, it became apparent I could combine the approach of Clarke, de Silva, and Thorley with the notion of a characteristic portfolio in Grinold and Kahn [2000] to build a framework that would address a wide range of attribution questions.

EX ANTE

I start with a few simple results from portfolio theory and show how many ex ante aspects of a portfolio can be captured as a covariance, variance, volatility, or correlation. This allows the gathering of several seemingly disparate aspects of the portfolio in a unified framework.[1]

I consider a collection of N assets and an N by N covariance matrix $\mathbf{V}$ whose n, m-th element, $V_{n,m}$, is the predicted covariance of the return to assets n and m. Risk predictions are annualized. A portfolio, say, portfolio P, is a vector of holdings $\mathbf{p} = \{p_1, p_2, \cdots p_N\}$ that does not have to sum to one or does not have to be non-negative. Thus we can represent long-short portfolios and the active portion of a long-only mandate as a portfolio.

The variance and risk of this portfolio are given, respectively, by:

$$\omega_{P,P} = \sum_{n=1,N} \sum_{m=1,N} p_n \cdot V_{n,m} \cdot p_m$$

$$\omega_P = \sqrt{\sum_{n=1,N} \sum_{m=1,N} p_n \cdot V_{n,m} \cdot p_m} \tag{1}$$

For two portfolios—call them X and Y—the covariance, $\omega_{X,Y}$, and correlation, $\rho_{X,Y}$, are given by:

$$\omega_{X,Y} = \sum_{n=1,N} \sum_{m=1,N} x_n \cdot V_{n,m} \cdot x_m$$

$$\omega_{X,Y} = \omega_X \cdot \rho_{X,Y} \cdot \omega_Y \tag{2}$$

At any time we have a forecast of annualized asset exceptional return that we call alpha: $\alpha = \{\alpha_1, \alpha_2 \cdots, \alpha_N\}$. The alpha of portfolio P is:

$$\alpha_P = \sum_{n=1,N} \alpha_n \cdot p_n \tag{3}$$

The information ratio of portfolio P, IR_P, is defined as the amount of alpha we expect to obtain in a year per unit of risk undertaken in pursuit of that alpha:

$$IR_P \equiv \frac{\alpha_P}{\omega_Q} \tag{4}$$

Our investment objective is to maximize what we call risk-adjusted or certainty-equivalent return:

$$U_P = \alpha_P - \frac{\lambda}{2} \cdot \omega_{P,P} \tag{5}$$

This is alpha less a penalty for risk. Of course, the overall objective must consider transaction costs and any limitations or constraints that might be imposed on the portfolio.

I start with the optimal solution to (5) in the absence of constraints or transaction costs. I'll call that *ideal* portfolio Q. One can obtain Q's holdings $\mathbf{q} = \{q_1, q_2 \cdots, q_N\}$ by solving the equations:

$$\alpha_n = \lambda \cdot \sum_{m=1,N} V_{n,m} \cdot q_m \tag{6}$$

where the sum $\sum_{m=1,N} V_{n,m} \cdot q_m$ is the covariance of asset n with portfolio Q. We write it as $\omega_{n,Q}$.

Thus (6) can be written as:

$$\alpha_n = \lambda \cdot \omega_{n,Q} \tag{7}$$

The alpha and risk-adjusted expected return of the ideal portfolio Q are, respectively:

$$\alpha_Q = \lambda \cdot \omega_{Q,Q}$$

and

$$U_Q = \frac{\lambda}{2} \cdot \omega_{Q,Q} \tag{8}$$

If we combine Equations (3) and (7), and recognize that $\sum_{n=1,N} p_n \cdot \omega_{n,Q} = \omega_{P,Q}$, we see that the alpha and information ratio for any portfolio P are given by:

$$\alpha_P = \lambda \cdot \omega_{P,Q} = \lambda \cdot \omega_Q \cdot \rho_{P,Q} \cdot \omega_P \tag{9}$$

After dividing Equation (9) through by ω_P we have:

$$IR_P = \lambda \cdot \omega_Q \cdot \rho_{P,Q} \tag{10}$$

The correlation, $\rho_{P,Q}$, of any portfolio P with the ideal portfolio Q is called the *transfer coefficient*. It is a measure of implementation efficiency.[2]

Notice that (10) implies:

- The information ratio of portfolio Q is equal to $\lambda \cdot \omega_Q$. This follows by setting $P = Q$ and noting $\rho_{Q,Q} = 1$.
- Portfolio Q has the highest possible information ratio because $\rho_{P,Q} \leq 1$. That highest information ratio is given by:

$$IR \equiv IR_Q = \lambda \cdot \omega_Q \tag{11}$$

Let portfolio B be the difference between P and Q. Portfolio B is called the *backlog*. It is the basket trade that we would need to make to take us from P to the ideal Q. The loss in objective value from holding P rather than Q is proportional to the variance of the backlog:

$$U_Q - U_P = \frac{\lambda}{2} \cdot \omega_{B,B} \tag{12}$$

What, one may ask, does this have to do with attribution? The key idea is that the risk, alpha, implementation efficiency, and loss in certainty-equivalent return can be expressed as proportional to, respectively, a risk, a covariance, a correlation, and a variance. And, as I show later, it is possible to model other aspects of any portfolio in terms of covariance and correlation as well. Thus we should turn our thinking toward a general ability to explain covariance and correlation if we want a flexible and unified notion of attribution.

WITH NO STRUCTURE

We start with little structure save the assets themselves. In this relatively simple environment, we can lay out the basic ideas that play a role in the more interesting and less transparent cases that follow.

Let X and Y represent two portfolios, each captured as an N element vector of possibly long and short holdings: $\mathbf{x} = \{x_1, x_2, \cdots, x_N\}$ and $\mathbf{y} = \{y_1, y_2, \cdots, y_N\}$. The covariance of portfolios X and Y is:

$$\omega_{X,Y} = \sum_{n=1,N} \sum_{m=1,N} x_n \cdot V_{n,m} \cdot y_m \tag{13}$$

We can rewrite this as:

$$\begin{aligned} \omega_{X,Y} &= \sum_{n=1,N} x_n \cdot \sum_{m=1,N} V_{n,m} \cdot y_m \\ &= \sum_{n=1,N} x_n \cdot \omega_{n,Y} \end{aligned} \tag{14}$$

where $\omega_{n,Y}$ is the covariance of asset n's return with the return on portfolio Y.

We can think of x_n as our exposure to asset n, and then of $\omega_{n,Y}$ as expressing the link between asset n and the portfolio Y. Note that this is correct at the margin as:

$$\frac{\partial \omega_{X,Y}}{\partial x_n} = \omega_{n,Y} \tag{15}$$

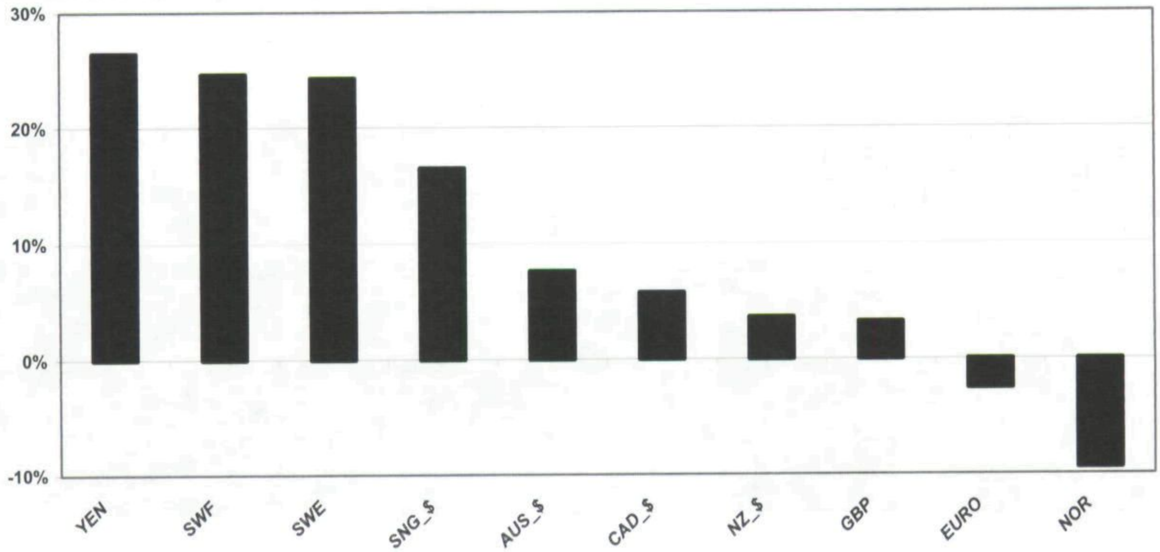
Of course, there is nothing special about either X or Y, so we could also write:

$$\omega_{X,Y} = \omega_{Y,X} = \sum_{n=1,N} y_n \cdot \omega_{n,X} \tag{16}$$

This seemingly innocent and ad hoc procedure of allocating the component $x_n \cdot \omega_{n,Y}$ (or $y_n \cdot \omega_{n,X}$) of $\omega_{X,Y}$ to asset n is exactly what we do in more structured (i.e., complicated) situations. There, under cover of a smidgen of theory and statistical technique, this procedure looks more elegant. Nevertheless, it is the same.

Exhibit 1

Contribution to Risk of Each Asset



Suppose X and Y are both equal to a portfolio P. Then the variance, $\omega_{P,P} = \omega_P^2$, and risk, ω_P, of portfolio P are given, respectively, by:

$$\omega_{P,P} = \sum_{n=1,N} \sum_{m=1,N} p_n \cdot V_{n,m} \cdot p_m$$

$$\omega_P = \sqrt{\sum_{n=1,N} \sum_{m=1,N} p_n \cdot V_{n,m} \cdot p_m} \tag{17}$$

Using the logic of Equation (2), we can write:

$$\omega_{P,P} = \sum_{n=1,N} p_n \cdot \omega_{n,P} \tag{18}$$

We attribute an amount of variance $P_n \cdot \omega_{n,P}$ to asset n and, similarly, a fraction of the variance, $P_n \cdot \omega_{n,P} / \omega_{P,P}$, to asset n.

Here is an example using a ten-asset long and short portfolio of ten major currencies.[3] The currencies are ordered so that the first currency, in our case the yen, makes up the greatest fraction of the budget.

If we wish to reduce portfolio risk, reducing the size of our active portfolio in the yen, the Swiss franc, or the Swedish krona would be a reasonable place to start. Note that both the Euro and the Norwegian krone are diversifying. Increasing our active positions in those assets reduces overall risk as they are acting as counterweight to other positions in the portfolio.

We'll look at two ways to interpret the allocation of risk in Equation (18). The first is based on correlation. Let's recall that we can break the covariance of asset n and portfolio P into components:

$$\omega_{n,P} = \omega_n \cdot \rho_{n,P} \cdot \omega_P \tag{19}$$

where ω_n is the risk of asset n, ω_P is the risk of portfolio P, and $\rho_{n,P}$ is the correlation of asset n and portfolio P.

If we define

$$\psi_{P,n} \equiv \frac{p_n \cdot \omega_n}{\omega_P} \tag{20}$$

the fraction of variance attributed to asset n can be written as:

$$\frac{p_n \cdot \omega_{n,P}}{\omega_{P,P}} = \psi_{P,n} \cdot \rho_{n,P}$$

where

$$\sum_{n=1,N} \psi_{p,n} \cdot \rho_{n,p} = 1 \tag{21}$$

We can think of $\psi_{P,n}$ as a standardized measure of portfolio P's exposure to asset n, and, of course, $\rho_{n,P}$ is the correlation of asset n with portfolio P. $\psi_{P,n}$ has many of the attributes of a correlation. Indeed, when all the N assets are uncorrelated, $\psi_{P,n}$ is equal to the correlation $\rho_{n,P}$.

The numbers behind Exhibit 1 are shown in Exhibit 2. The long-short holdings of each asset are in column 1.

EXHIBIT 2

Risk Attribution for Currency Portfolio

	p(*n*)	rho(*n*,*P*)	psi(*P*,*n*)	product
YEN	-9.62%	-0.554	-0.477	26.44%
SWF	-10.47%	-0.472	-0.522	24.62%
SWE	-12.14%	-0.425	-0.570	24.25%
SNG_$	-11.65%	-0.435	-0.379	16.48%
AUS_$	8.54%	0.184	0.413	7.60%
CAD_$	9.38%	0.175	0.327	5.74%
NZ_$	5.39%	0.132	0.272	3.60%
GBP	-2.45%	-0.315	-0.102	3.22%
EURO	1.40%	-0.378	0.067	-2.53%
NOR	12.94%	-0.156	0.606	-9.43%

The holding in the U.S. dollar is minus the sum of these holdings, about 8% in this case. The second column indicates the correlation of each asset with the overall portfolio, $\rho_{n,P}$. The third column shows the derived exposure $\psi_{P,n}$ and the last column the product $\psi_{P,n} \cdot \rho_{n,P}$ (these products sum to one).

Note the apparent paradox that increasing the largest position, the Norwegian krone, actually reduces risk. The source is readily identified. We are short about 25% in European currency, and the long euro and Norwegian krone positions are not enough to counterbalance.

STRUCTURE

We can attribute the risk, alpha, information ratio, transfer coefficient, and loss in utility of any portfolio P to specific sources other than the assets themselves.[4] In a departure from the norm, we can think of these sources as portfolios rather than asset groups such as sectors or common factors such as value, size, and momentum. The idea is to take abstract concepts such as liquidity and growth and give them an investment persona. Then we can answer questions about the relationship between our portfolio and liquidity and growth by examining the expected return correlations between our portfolio and the liquidity and growth portfolios.

Let S_j; $j = 1, \cdots, J$ be J possible sources. A portfolio represents each source. The holdings in these portfolios are $\mathbf{s}'_j = \{s_1, s_{2,j}, \cdots, s_{N,j}\}$ where $s_{n,j}$ is the holding of asset n in source portfolio j. Each source, as a portfolio, has its easily calculated predicted risk, alpha, and information ratio. The predicted covariance and correlation between any two sources is calculated in the usual manner:

EXHIBIT 3

Significant Properties of Portfolios

position	risk	alpha	IR
FAST	4.00%	4.00%	1.00
INT	3.00%	2.25%	0.75
SLOW	2.00%	1.00%	0.50
Q	6.65%	11.07%	1.66
P	4.77%	5.81%	1.22

EXHIBIT 4

Return Correlations of Portfolios and Source Portfolios

correlations	FAST	INT	SLOW
rho<*j*,*Q*>	0.601	0.451	0.301
rho<*j*,*P*>	0.231	0.584	0.536

$$\omega_{j,k} = \sum_{n=1,N} \sum_{m=1,N} s_{n,j} \cdot V_{n,m} \cdot s_{m,k}$$

$$\omega_{j,k} = \omega_j \cdot \rho_{j,k} \cdot \omega_k \tag{22}$$

I make consistent use of a representative example to illustrate the points. In this example I partition the alpha sources into three groups according to their rate of change: FAST for the signals that change quickly, SLOW for the signals that change slowly, and INT for those somewhat in between. There are also an ideal portfolio Q and an actual portfolio P that falls short of the ideal. The forecasted risk, alpha, and IR of these five portfolios in our example are given in Exhibit 3. The correlations between the portfolios Q and P and the three source portfolios are given in Exhibit 4.

The results are in line with the general thrust of the example. Portfolio Q, unencumbered by trading costs, pursues the stronger FAST source aggressively and pays less attention to the SLOW and INT sources. The actual implementation is deterred by transaction costs so it moves more slowly. Portfolio P has a higher correlation with the slower moving signals. The line $\rho_{j,Q}$ is of special interest because these are the transfer coefficients of the source portfolios.

To analyze any portfolio, we use a regression to parse the aspects of the holdings that are explained by each of the sources. The regression for a generic portfolio X is:

$$x_n = \sum_{j=1,J} s_{n,j} \cdot \beta_{X,j} + \varepsilon_{X,n} \tag{23}$$

The regression splits the portfolio X into a component—let's call it $\hat{x}_n = \sum_{j=1,J} s_{n,j} \cdot \beta_{X,j}$ —that is explained by the sources and a residual portfolio, $\boldsymbol{\varepsilon}'_X = \{\varepsilon_{X,1}, \varepsilon_{X,2}, \cdots, \varepsilon_{X,N}\}$, that is not explained by the sources. The regression is designed so that the residual and each of the source portfolios are uncorrelated:

$$0 = \sum_{n=1,N} \sum_{m=1,N} s_{n,j} \cdot V_{n,m} \cdot \varepsilon_{X,m} \tag{24}$$

for $j = 1, 2, \ldots, J$.

The regression results for portfolios P and Q are shown in Exhibit 5. In the example the ideal portfolio Q is a mixture of the three sources, so the regression R^2 is one. For the actual portfolio P, the regression can explain only 87.34% of the portfolio's risk.

Look again at regression Equation (23). The residual portion is, as I note in Equation (24), uncorrelated with all the sources. We make frequent use of the relationship between the risk of X, the risk of the residual $\varepsilon(X)$, and the correlation between the two as captured in Equation (25):

$$\omega_{\varepsilon(X)} = \omega_X \cdot \rho_{X,\varepsilon(X)} \tag{25}$$

where $\omega_{\varepsilon(X)}$ is the risk of the residual portfolio ε_X, ω_X is the risk of portfolio X, and $\rho_{X,\varepsilon(X)}$ is the correlation between X and its residual ε_X.

In the example, $\rho_{P,\varepsilon(P)} = 0.355$. We can think of the regression R^2 as:[5]

$$\sqrt{1 - \rho^2_{X,\varepsilon(X)}}$$

Within this framework the covariance of any two portfolios X and Y can be written as:

$$\omega_{X,Y} = \sum_{j=1,J} \beta_{X,j} \cdot \omega_{j,Y} + \omega_{\varepsilon(X),\varepsilon(Y)} \tag{26}$$

where $\beta_{X,j}$ is the regression coefficient for portfolio X and source j, $\omega_{j,Y}$ is the covariance between portfolio Y and source j, and $\omega_{\varepsilon,(X),\varepsilon(Y)}$ is the covariance between the residual portfolios of X and Y. The component $\sum_{j=1,J} \beta_{X,j} \cdot \omega_{j,Y}$ is the fraction of covariance explained by the sources, and the component, $\omega_{\varepsilon,(X),\varepsilon(Y)}$, is the covariance of the residuals and consequently not explained by the sources.

We can, again as I demonstrate in the appendix, go a step farther and attribute the *correlation* between X and Y to the sources:

EXHIBIT 5

Regression Coefficients and R^2

beta	FAST	INT	SLOW	R-sqrd
beta<j,Q>	1.58	1.22	1.99	100.00%
beta<j,P>	0.75	0.96	1.66	87.34%

$$\rho_{X,Y} = \sum_{j=1,J} \psi_{X,j} \cdot \rho_{j,Y} + \rho_{X,\varepsilon(X)} \cdot \rho_{\varepsilon(X),Y} \tag{27}$$

In Equation (27):

$$\psi_{X,j} \equiv \frac{\beta_{X,j} \cdot \omega_j}{\omega_X} \tag{28}$$

is a standardized version of the regression coefficient that adjusts for the different levels of risk in source portfolio j and portfolio X. If we rescale the source portfolios so they all have the same risk as portfolio X, $\psi_{X,j}$ would be the resulting regression coefficient.

We can also think of $\psi_{X,j}$ as a *pseudo-correlation*. Note that it is dimensionless, as $\beta_{X,j}$ is in units of X per unit of j, and that it is indeed a correlation in the special case where the sources are uncorrelated; i.e., $\omega_{j,k} = 0$ if $j \neq k$.

Notice also that the risk of the component $\beta_{p,j} \cdot \mathbf{s}_j$ of portfolio P is $\omega_j \cdot \beta_{p,j} = \omega_P \cdot \psi_{p,j}$. Thus, at least in isolation, we can consider this product a risk allocation to source j.

Risk

We can use Equation (27) to attribute the variance of portfolios Q and P to the three sources. When $X = Y$, the sum in (27) is necessarily equal to one, so we can account for all the risk.

The breakdown in the example is shown in Exhibit 6. The numbers in each row sum to one. The risk in portfolio Q is 100% explained by the sources, so the residual is 0.0. For the actual portfolio P, 12.66% is not explained by the sources. Once again we can see how transaction costs slow down an actual implementation. For Q, 57.16% of the risk is explained by the FAST signal, while that accounts for slightly less than 15% of the risk in the actual implementation.

Alpha and the Information Ratio

If we are trying to explain portfolio alpha, we know from our analysis that:

EXHIBIT 6
Risk Budgets for Ideal and Managed Portfolios

Risk Budget	FAST	SLOW	INT	Residual
Q	57.16%	24.86%	17.98%	0.00%
P	14.59%	35.41%	37.34%	12.66%

$$\alpha_P = \lambda \cdot \omega_{P,Q} = \lambda \cdot \omega_Q \cdot \rho_{P,Q} \cdot \omega_P = IR \cdot \omega_P \cdot \rho_{P,Q} \quad (29)$$

In this breakdown the portfolio alpha, α_P, depends on three things:

- The environment in terms of the overall potential of the alphas captured by *IR*.
- The level of risk in the portfolio ω_P.
- The transfer coefficient $\rho_{P,Q}$.

We can apply Equation (29) to each source. For source *j*, we obtain:

$$\begin{aligned}\alpha_j &= \lambda \cdot \omega_{j,Q} \\ &= \lambda \cdot \omega_Q \cdot \rho_{j,Q} \cdot \omega_j \\ &= IR \cdot \omega_j \cdot \rho_{j,Q} \end{aligned} \quad (30)$$

To go farther, we need to look in detail at the transfer coefficient, $\rho_{P,Q}$. The transfer coefficient can be attributed to three sources and a residual as per Equation (27):

$$\alpha_P = IR \cdot \omega_P \cdot \left\{ \sum_{j=1,J} \psi_{P,j} \cdot \rho_{j,Q} + \rho_{P,\varepsilon(P)} \cdot \rho_{\varepsilon(P),Q} \right\} \quad (31)$$

The part attributed to source *j* is $\psi_{P,j} \cdot \rho_{j,Q}$, where $\rho_{j,Q}$ is the transfer coefficient of source *j*, and $\psi_{P,j}$ is the standardized exposure of portfolio *P* to source *j*.

Exhibit 7 describes the breakdown for portfolio *P*. The first panel tells what is possible. The maximum information ratio, $IR = IR_Q$, multiplied by the risk of the portfolio tells us that the best we could hope for (transfer coefficient of one) is an alpha of 7.94% from the product of *IR* and ω_P.

The next sections of Exhibit 7 detail the alpha contributions by source. For source *j*, the contribution is $IR \cdot \omega_P \cdot \{\psi_{p,j} \cdot \rho_{j,Q}\}$, where $IR \cdot \omega_P$ is the potential, $\psi_{p,j}$ the standardized exposure, and $\rho_{j,Q}$ the transfer coefficient of source *j*. For example, our exposure to the SLOW source is $\psi_{P,\,SLOW} = 0.097$, and the transfer coefficient of the SLOW source is $\rho_{SLOW,Q} = 0.301$. Thus we attribute $7.94\% \cdot \{0.697 \cdot 0.301\} = 1.66\%$ to the SLOW source.

EXHIBIT 7
Analysis of Alpha for Portfolio *P*

Alpha Report			
portfolio	*P*		
IR	1.66		
risk_*P*	4.77%		
Source	FAST	INT	SLOW
expo: psi<*P*,*j*>	0.631	0.606	0.697
risk<j>	3.01%	2.89%	3.33%
TC: rho<*j*,*Q*>	0.601	0.451	0.301
*IR**risk**TC*	3.01%	2.17%	1.66%
source alpha	6.85%		
Residual			
expo: rho<*P*,e(*P*)>	0.356		
residual risk <Q>	1.70%		
TC: rho<e(*P*),*Q*>	0		
residual alpha	0.00%		
Totals			
portfolio Alpha	6.85%		
portfolio *TC*	0.862		

Loss in Value-Added

We can switch from the glass is two-thirds full perspective and move to the glass is one-third empty, and ask how much of our expected value-added we are giving up by holding *P* rather than *Q*. Recall our previous result that the loss in value-added is proportional to the variance of the backlog: $B = Q - P$:

$$U_Q - U_P = \frac{\lambda}{2} \cdot \omega_{B,B} \quad (32)$$

In the example, $\lambda = 25$, so we must explain the variance of the backlog as per Equation (6) with both *X* and *Y* equal to *B*. The backlog variance in Exhibit 8 occurs predominantly because we are behind the curve on the FAST signal and the residual risk in portfolio *P*. The expected loss in terms of risk-adjusted return is 1.54%.

EX POST

The goal is a unified and portfolio-centered view of both ex post and ex ante attribution. This unified perspective allows us to approach both aspects in the same way. We can compare expectations with realizations, and sort out what worked, what didn't, and why.

This pleasing symmetry has a price; we must represent the returns with a portfolio. This is done in exactly the same fashion used in the ex ante case. Ex ante, we

Exhibit 8

Analysis of Utility Loss

Backlog	FAST	INT	SLOW	Residual	Total
util loss	1.20%	0.02%	-0.05%	0.36%	1.54%
rho<B,B>	0.781	0.013	-0.030	0.235	1.00

represented the forecasts as an ideal portfolio. Ex post, we use hindsight to represent the returns as an ex post ideal portfolio. After we have made that small leap, the ex post attribution proceeds in a manner that is analogous to the ex ante cases discussed so far.

This analysis has two interesting by-products: the notion of the opportunity set as an ex post analogue to the information ratio and the notion of a realized information coefficient (IC) as the ex post analogue of the transfer coefficient.

Ex Post Ideal

We let θ_n be the return on asset n. Suppose that our forecast of alpha is a spot-on θ_n. What portfolio would we have built with those perfect forecasts? Call that retrospective ideal portfolio R. Portfolio R has holdings $\mathbf{r} = \{r_1, r_2, \cdots, r_N\}$ that are calculated in the same manner as the holdings of the ex ante ideal portfolio Q:[6]

$$\theta_n = \lambda \cdot \sum_{n=1,N} V_{n,m} \cdot r_m \tag{33}$$

As before, we can write Equation (33) as:

$$\theta_n = \lambda \cdot \omega_{n,R} = \lambda \cdot \omega_R \cdot \rho_{n,R} \cdot \omega_n \tag{34}$$

We can then aggregate (34) and represent the return on any portfolio P in terms of its covariance with R:

$$\begin{aligned} \theta_P &= \sum_{n=1,N} \theta_n \cdot p_n = \lambda \cdot \omega_{P,R} \\ &= \lambda \cdot \omega_R \cdot \omega_P \cdot \rho_{P,R} \end{aligned} \tag{35}$$

Opportunity Set and Realized IC

From Equation (35) we see that the realized return on any portfolio P is proportional to its risk, ω_P, and its correlation with the ex post ideal $\rho_{Q,R}$. Equation (3) tells us three things:

- The ratio θ_R/ω_R equals $\lambda \cdot \omega_R$ because $\rho_{R,R} = 1$.
- The ratio θ_P/ω_P for any portfolio P is less than $\lambda \cdot \omega_R$ because $\rho_{P,R} \leq 1$.
- The correlation $\rho_{P,R}$ is the ex post analogue of the transfer coefficient $\rho_{P,Q}$.

We'll call $\rho_{P,R}$ the *realized information coefficient* or realized IC of portfolio P. The maximum ratio of realized return to predicted risk that can be obtained is called the *opportunity set*.[7] It is:

$$OS = \frac{\theta_R}{\omega_R} = \lambda \cdot \omega_R \tag{36}$$

Thus, explaining performance for portfolio P is equivalent to explaining the correlation $\rho_{P,R}$ because:

$$\theta_P = OS \cdot \omega_P \cdot \rho_{P,R} \tag{37}$$

The product in Equation (37) neatly partitions the return into three buckets: OS, representing the maximum opportunity available; ω_P, representing the portfolio's aggressiveness; and the realized IC, $\rho_{P,R}$, which measures how effective we are in aligning our portfolio with the ex post ideal R.

For the source portfolios S_j; $j = 1, J$, we have:

$$\theta_j = OS \cdot \omega_j \cdot \rho_{j,R} \tag{38}$$

The correlation $\rho_{j,R}$ is the realized IC for source j. It tells us in a standardized way how well our forecasts have done in this particular case. If we put Equation (37) together with our analysis of correlation, Equation (27), we have the ex post analogue of Equation (31):

$$\theta_P = OS \cdot \omega_P \cdot \left\{ \sum_{j=1,J} \psi_{P,j} \cdot \rho_{j,R} + \rho_{P,\varepsilon(P)} \cdot \rho_{\varepsilon(P),R} \right\} \tag{39}$$

Let's return to the example and see how this might work. In our case, we need only five pieces of data for the ex post analysis: the size of the opportunity set OS; the realized information coefficients of the three source portfolios, $\rho_{j,R}$; and the correlation between the residual component of portfolio P and the return portfolio R, $\rho_{\varepsilon(P),R}$. In this example, using monthly residual returns, we have a typical value $OS = 10.51$. The correlation of the residual component of P with the return component is $\rho_{\varepsilon(P),R} = -0.117$ so we lose on the residual.

Exhibit 9 gives the performance for the three sources. As per Equation (38), the realized IC row is called rho <j, R>. As we can see, the FAST and SLOW signals were successful as predictions. Thus the 3.66%

EXHIBIT 9

Performance of Source Portfolios

Source	FAST	INT	SLOW
risk (omega)	4.00%	3.00%	2.00%
realized *IC* (rho)	0.087	-0.045	0.096
return (theta)	3.66%	-1.42%	2.02%

return attributed to the FAST source is the product of the opportunity set, 10.51, the risk of the FAST source, 4.00%, and the realized IC of the fast source, 0.087. The intermediate horizon source, INT, was less fortunate. An investment in source INT would lose 1.42%.

With this structure we can analyze the return on our portfolio in exactly the manner used to examine the properties of the alpha forecasts in Exhibit 7. The opportunity set is 10.51. The risk of the portfolio is 4.77%, so the potential return capture is an immense 50.18%.

This is not an atypical number. This huge opportunity is complemented by our tiny ability to seize it. We must summon a tad of humility and realize that capturing 1/20 of the potential would be an outstanding outcome.

The standardized exposures to the three return sources are exactly the same as the standardized exposures to the three alpha sources. The ex ante and ex post exposures agree. As we indicate above, the FAST and SLOW signals are successful, while the INT is not.

For the slow signal, the exposure is 0.697, and the realized IC (as we see in Exhibits 9 and 10) is 0.096. Thus we attribute a fraction $0.697 \times 0.096 = 0.067$ of the total potential return to the SLOW source. In this case, the total potential is 50.18%, so we attribute $0.067 \times 50.18\% = 3.36\%$ to the SLOW source. In the same manner we attribute fractions to the FAST and INT sources, and find a net contribution from the sources of 4.75%.

The residual exposure is $\rho_{P,\varepsilon(P)} = 0.356$. This means our residual risk is $\omega_P \cdot \rho_{P,\varepsilon(P)} = 4.77\% \times 0.356 = 1.70\%$. As it turns out, this residual portfolio has a realized IC of $\rho_{\varepsilon(P),R} = -0.117$, so we lose $10.51 \cdot \{1.70\%\} \cdot \{-0.117\} = 2.09\%$ due to our residual position. All in, we are ahead, $4.75\% - 2.09\% = 2.66\%$. The realized IC for the entire portfolio is 0.06.

We can perform the same analysis for the ideal portfolio Q and the backlog B. In the first case, we are asking how the model performed, and in the second we are asking how our implementation may have influenced performance.

EXHIBIT 10

Ex Post Analysis of Portfolio *P*

Return Report			
portfolio	*P*		
OS	10.51		
risk_*P*	4.77%		
Source	FAST	INT.	SLOW
expo: psi<*P*,*j*>	0.631	0.606	0.697
risk<j>	3.01%	2.89%	3.33%
IC: rho<*j*,*R*>	0.087	-0.045	0.096
*OS**risk**IC*	2.76%	-1.37%	3.36%
source return	4.75%		
Residual			
expo: rho<*P*,e(*P*)>	0.356		
residual risk <Q>	1.70%		
IC: rho<e(*P*),*R*>	-0.117		
OS*risk**IC*	-2.09%		
Totals			
portfolio return	2.66%		
portfolio *IC*	0.053		

ADDITIONAL POINTS

Further elements include risk control, vintage of alpha, and industry practice.

Risk Control

Our discussion of ex post attribution concentrates on the element of return, θ_n, that we are attempting to forecast. We also want to control risk in the other components of return. Let's say that the total excess return on asset n is $\chi_n = \phi_n + \theta_n$, where θ_n is the component of return where we hope to add value, and ϕ_n is the part of the return that we wish to control.

For example, in an industry-neutral strategy, ϕ_n would be the industry return. The appendix indicates how one can separate these two elements of return. The total return to portfolio P is:

$$\chi_P = \phi_P + \theta_P$$
$$\phi_P = \sum_{n=1,N} \phi_n \cdot p_m$$
$$\theta_P = \sum_{n=1,N} \theta_n \cdot p_m \tag{40}$$

We can identify sources of risk, portfolios $\hat{\mathbf{s}}_k$ $k = 1, ..., K$, not to be confused with the sources of alpha. To continue the example, these K sources could be K industry portfolios.[8] We can analyze portfolio P with respect to those sources of risk in exactly the same way as analyzing it against the alpha sources; recall Equation (23):

$$p_n = \sum_{k=1,K} \hat{s}_{n,k} \cdot \hat{\beta}_{P,k} + \hat{\varepsilon}_{P,n} \tag{41}$$

The exposures $\hat{\beta}_{P,k}$ and residual $\hat{\varepsilon}_P$ are of course distinct from the exposures and residuals that we obtained with the regression against alpha sources.

We then build a risk portfolio F with holdings $\mathbf{f} = \{f_1, f_2, \cdots, f_N\}$ determined by:

$$\phi_n = \lambda \cdot \sum_{n=1,N} V_{n,m} \cdot f_m \tag{42}$$

Then, following the logic of the results so far, the risk-controlled component of return for portfolio P is:

$$\begin{aligned} \phi_P &= \lambda \cdot \omega_{F,P} \\ &= \lambda \cdot \omega_F \cdot \rho_{F,P} \cdot \omega_P \end{aligned} \tag{43}$$

Portfolio F has the maximum ratio of this controlled return per unit of predicted risk. That ratio is $CR = \lambda \cdot \omega_F$. Thus, we can write:

$$\begin{aligned} \phi_P &= \lambda \cdot \omega_{F,P} \\ &= CR \cdot \omega_P \cdot \left\{ \sum_{k=1,K} \hat{\psi}_{P,k} \cdot \rho_{k,F} + \rho_{P,e(P)} \cdot \hat{\rho}_{e(P),F} \right\} \end{aligned} \tag{44}$$

where, to distinguish them from exposures to alpha sources, I have placed hats over the standardized exposures, $\hat{\psi}_{P,k}$, to the sources of risk. Note you could do much of the same analysis of the alpha sources themselves along with the ex ante ideal Q and the residual e(P) of portfolio P to the alpha source.

Indeed, the flexibility one has in this regard is both a blessing and a curse: a curse because the ability to disgorge a mass of data can obscure understanding as much as help it. The great flexibility means that questions should be thought through in advance, and priors established, and then the attribution scheme should be set up in light of those questions and priors.

Purists may recoil at the suggestion of one attribution for our pursuit of alpha and another for our efforts at risk control. We instead take to heart the proverb that the hunter who chases two rabbits won't catch either.

Vintage Alpha Sources

The example to this point considers alpha by source either a fast signal, a slow signal, or an intermediate signal, or a value signal, a growth signal, or a sentiment signal. It is also possible to analyze alpha along the time dimension, i.e., by date of arrival. At time t we have N alphas $\alpha_n(t)$. In addition, we know the alphas that were current for those N assets at earlier times $\alpha_n(t-\tau)$ for $\tau = 1, 2 \ldots, T$.[9] We can consider the portfolios $Q(t-\tau)$ with holdings $\mathbf{q}(t-\tau)$ that we would hold today based on the information of τ periods ago.[10] It is possible to separate today's ideal holding $\mathbf{q}(t)$ into (see the appendix for details):

- An initial position $\mathbf{q}(t-T)$ where $\mathbf{q}(t-T)$ is the portfolio we would hold today based on all the information available at time $t-T$.
- And T increments, $\mathbf{u}(t-\tau)$, where $\mathbf{u}(t-\tau)$ is the portfolio we would hold today based solely on the new information that came in from $t-(\tau+1)$ to $(t-\tau)$.

If, for example, we are considering monthly periods over a year, T is 12. There are 12 increments of information and a portfolio $\mathbf{q}(t-12)$ that we would ideally hold today if we were using the information, $\alpha(t-12)$, that was available one year ago. The base position, $\mathbf{q}(t-12)$, and the 12 increments, $\mathbf{u}(t-\tau)\tau = 0, 1, \ldots, 11$, can be used as the sources. They will by construction explain 100% of the current ideal $\mathbf{q}(t)$.

The ex ante analysis tells us how exposed we are to information sorted by its date of arrival. The ex post analysis tells us how well we fared in each time segment. We can compare the risk budget of the ideal, $\mathbf{q}(t)$, with any actual portfolio to gauge our exposure to dated rather than current information.

X, Y or *Y, X*

We have shown how to model ex ante alpha and ex post returns (among other things) as proportional to a covariance or correlation. With the introduction of various sources, we can decompose the correlation into a sum of standardized exposures of one portfolio to the sources multiplied by correlations of the other portfolio with the sources. The exposures are multivariate; they come from the regression. The transfer coefficients and realized ICs are bivariate; they are the correlations between two portfolios. We have chosen to do this so that we can interpret the correlations as transfer coefficients ex ante and realized information coefficients ex post, while the exposures are the same in both cases.

We could have chosen to go the other route, and model the standardized exposures to the sources of the ex ante ideal Q and ex post ideal R, and used the correlations of actual portfolio P. For alpha, this yields:

$$\alpha_P = IR \cdot \omega_P \cdot \left\{ \sum_{j=1,J} \psi_{Q,j} \cdot \rho_{j,P} + \rho_{P,e(P)} \cdot \rho_{e(P),Q} \right\} \tag{45}$$

Compare Equation (45) with Equation (31). For ex post return, we get:

$$\theta_P = OS \cdot \omega_P \cdot \left\{ \sum_{j=1,J} \psi_{R,j} \cdot \rho_{j,P} + \rho_{P,e(P)} \cdot \rho_{e(P),R} \right\} \tag{46}$$

Compare Equation (46) with Equation (39).

A most interesting comparison is when P is one of the source portfolios, say, S_ℓ. Then (45) and (46) become:

$$\alpha_\ell = IR \cdot \omega_\ell \cdot \left\{ \sum_{j=1,J} \psi_{Q,j} \cdot \rho_{j,\ell} \right\}$$

and

$$\theta_\ell = OS \cdot \omega_\ell \cdot \left\{ \sum_{j=1,J} \psi_{R,j} \cdot \rho_{j,\ell} \right\} \tag{47}$$

If we use (31), we have what appears to be a simpler and more reasonable result. The exposures and standardized exposures to a source S_ℓ are $\beta_{\ell,j} = \psi_{\ell,j} = 0$ if $j \neq \ell$ and $\beta_{\ell,\ell} = \psi_{\ell,\ell} = 1$. Therefore we have expressions:

$$\alpha_\ell = IR \cdot \omega_\ell \cdot \rho_{\ell,Q}$$

and

$$\theta_\ell = OS \cdot \omega_\ell \cdot \rho_{\ell,R} \tag{48}$$

that rely heavily on the transfer coefficient $\rho_{\ell,Q}$ and realized information coefficient $\rho_{\ell,R}$.

Industry Practice

I know of one venerable and widely used ex post attribution system based on returns regression. We start with a group of K factors representing industries and common characteristics such as size, valuation, liquidity, and volatility. The factors are captured in an N by K matrix with coefficients $x_{n,k}$. We associate returns with the factors using a generalized least squares regression of the asset returns θ_n against the factors:

$$\theta_n = \sum_{k=1,K} x_{n,k} \cdot f_k + u_n \tag{49}$$

The estimated coefficients f_k are called factor returns and can be interpreted as the return on the portfolio.

Given a portfolio P with holdings $\mathbf{p} = \{p_1, p_2, \cdots, p_N\}$, we calculate the exposures as:

$$x_{P,k} \equiv \sum_{n=1,N} p_n \cdot x_{n,k} \tag{50}$$

of portfolio P to common factor k. The return on portfolio P is then:

$$\theta_P = \sum_{k=1,K} x_{P,k} \cdot f_k + \sum_{n=1,N} p_n \cdot u_n \tag{51}$$

We attribute the amount $x_{P,k} \cdot f_k$ (exposure $x_{P,k}$ times factor return f_k) to factor k. The remainder, $\sum_{n=1,N} p_n \cdot u_n$, is residual.

There is another way to interpret this procedure and link it with the method advanced here. For each of the K factors, we can associate a source portfolio S_k with holdings $s_{k,n}$ by solving the equations:

$$x_{n,k} = \lambda \cdot \sum_{m=1,N} V_{n,m} \cdot s_{k,m} = \lambda \cdot \omega_{n,k} \tag{52}$$

In this context we can think of portfolio P's exposures as [see Equation (50)]:

$$x_{P,k} = \lambda \cdot \omega_{P,k} = \lambda \cdot \omega_k \cdot \rho_{k,P} \cdot \omega_P \tag{53}$$

Thus the factor exposure is the risk aversion times a covariance of the source portfolio and the portfolio P that we wish to analyze. If we regress the ex post ideal portfolio R as in Equation (23) against the source portfolios:

$$r_n = \sum_{k=1,K} s_{n,k} \cdot \beta_{R,k} + \varepsilon_{R(n)} \tag{54}$$

we find, as is demonstrated in the appendix, that the resulting estimates, $\beta_{R,k}$ are exactly equal to the factor returns f_k obtained from Equation (49). Thus:

$$f_k = \beta_{R,k} = \frac{\omega_R}{\omega_k} \cdot \psi_{R,k} \tag{55}$$

where $\psi_{R,j}$ is the exposure adjusted for different risk levels. Therefore, the return attributed to factor (source) k can be written in several ways:

$$\begin{aligned} x_{P,k} \cdot f_k &= x_{P,k} \cdot \beta_{R,k} \\ &= \lambda \cdot \omega_R \cdot \omega_P \cdot \psi_{R,k} \cdot \rho_{k,P} \\ &= OS \cdot \omega_P \cdot \psi_{R,k} \cdot \rho_{k,P} \end{aligned} \tag{56}$$

where $OS = \lambda \cdot \omega_R$ is the size of the opportunity set.

If we compare this with Equation (46), we see it is just the Y, X method of splitting up the covariance rather than the X, Y method I have suggested.

Another Perspective

The risk of the residual component of a portfolio is $\omega_P \cdot \rho_{P,\varepsilon(P)}$. Also, the risk of the j-th source portfolio times its exposure $\beta_{P,j}$ is $\omega_j \cdot \beta_{P,j} = \omega_P \cdot \psi_{P,j}$. We can think of these products as risk allocations as in the formulas for portfolio risk, portfolio alpha, and portfolio return:

$$\omega_P = 1 \cdot \left\{ \sum_{j=1,J} \left(\omega_P \cdot \psi_{P,j}\right) \cdot \rho_{j,P} + \left(\omega_P \cdot \rho_{P,\varepsilon(P)}\right) \cdot \rho_{\varepsilon(P),P} \right\}$$

$$\alpha_P = IR \cdot \left\{ \sum_{j=1,J} \left(\omega_P \cdot \psi_{P,j}\right) \cdot \rho_{j,Q} + \left(\omega_P \cdot \rho_{P,\varepsilon(P)}\right) \cdot \rho_{\varepsilon(P),Q} \right\}$$

$$\theta_P = OS \cdot \left\{ \sum_{j=1,J} \left(\omega_P \cdot \psi_{P,j}\right) \cdot \rho_{j,R} + \left(\omega_P \cdot \rho_{P,\varepsilon(P)}\right) \cdot \rho_{\varepsilon(P),R} \right\} \quad (57)$$

We see that the risk from each source and the risk from the residual are consistently represented in the equations allocating portfolio risk, portfolio alpha, and portfolio return. The whole *and each part* can be considered the product of three entities:

- A constant that is independent of the portfolio or part; for example, 1, IR or OS.
- A stand-alone risk; for example, ω_p for the whole, $\omega_p \cdot \psi_{P,j}$ for source j, or $\omega_p \cdot \rho_{P,\varepsilon(P)}$ for the residual.
- A correlation such as $\rho_{P,R}$ for the whole, $\rho_{j,R}$ for source j, and $\rho_{\varepsilon(P),R}$ for the residual.

CONCLUSION

I have presented a flexible, unified, and portfolio-centered approach to attribution. The key to the analysis is framing the relevant question in portfolio terms. This perspective allows attribution by examination of the covariance or correlation of return between two portfolios. I demonstrated how to use this when there is little structure and when it is easy to identify possible sources of expected return (alpha) and risk.

The same procedure that allows an ex ante analysis of a portfolio's alpha and predicted risk could be used ex post to look at returns in much the same format. For ex ante analysis, the procedure highlights the role of the transfer coefficient as the correlation of any portfolio's predicted return with the predicted return on an ideal portfolio. For ex post analysis, we use an ex post ideal portfolio and refer to the correlation of any portfolio's return with the return on that ex post ideal as the realized information coefficient; the realized IC is the ex post analogue we of the transfer coefficient.

My approach and more traditional methods are in a sense equivalent; both attribute return (or alpha) to a source through a notion of exposure times result. In the traditional return regression models, the exposure to a source is bivariate and based on the relationship between that source and the portfolio analyzed, while the results (factor returns) are derived through a multivariate (generally regression) analysis. For the methods I suggest, the exposures are obtained in a multivariate manner, and the results are bivariate.

APPENDIX

Explanations and Derivations

This appendix provides support for assertions made in the text that are more technical in nature. There are five parts:

- Description of the portfolio regression Equation(23).
- Derivation of Equation (27).
- Separation of the risk-controlled and forecast returns as per Equation (40).
- Determining the new information flows in each period as discussed in the vintage example.
- Equivalence of the return regression Equation (49) and the portfolio regression Equation (54).

Portfolio Regression

Equation (23), in matrix notation, is:

$$\mathbf{X} = \mathbf{S} \cdot \boldsymbol{\beta}_x + \boldsymbol{\varepsilon}_x \quad \text{(A-1)}$$

If we premultiply this by $\mathbf{S}' \cdot \mathbf{V}$, we get:

$$\mathbf{S}' \cdot \mathbf{V} \cdot \mathbf{x} = \left\{\mathbf{S}' \cdot \mathbf{V} \cdot \mathbf{S}\right\} \cdot \boldsymbol{\beta}_x + \mathbf{S}' \cdot \mathbf{V} \cdot \boldsymbol{\varepsilon}_x \quad \text{(A-2)}$$

The goal is that the residual be uncorrelated with the sources. Thus $\mathbf{S}' \cdot \mathbf{V} \cdot \boldsymbol{\varepsilon}_X = \mathbf{0}$, and we have:

$$\boldsymbol{\beta}_x = \left\{\mathbf{S}' \cdot \mathbf{V} \cdot \mathbf{S}\right\}^{-1} \cdot \mathbf{S}' \cdot \mathbf{V} \cdot \mathbf{x} \quad \text{(A-3)}$$

Note that $\{\mathbf{S}' \cdot \mathbf{V} \cdot \mathbf{S}\}$ is the J by J covariance of the source portfolios.

The risk of the residual holdings is:

$$U_Q - U_P = \frac{\lambda}{2} \cdot \omega_{B,B} \tag{A-4}$$

For any portfolio Y, the covariance with this residual is:

$$\omega_{Y,\varepsilon(X)} = \mathbf{y}' \cdot \mathbf{V} \cdot \boldsymbol{\varepsilon}_x = \boldsymbol{\omega}_Y \cdot \rho_{Y,\varepsilon(X)} \cdot \boldsymbol{\omega}_{\varepsilon(X)} \tag{A-5}$$

Since $\mathbf{x}' \cdot \mathbf{V} \cdot \boldsymbol{\varepsilon}_X = \boldsymbol{\varepsilon}'_X \cdot \mathbf{V} \cdot \boldsymbol{\varepsilon}_X$, we can easily establish that:

$$\omega_X \cdot \rho_{X,\varepsilon(X)} = \omega_{\varepsilon(X)} \tag{A-6}$$

Derivation of Equation (27)

We start with Equation (26):

$$\omega_{X,Y} = \sum_{j=1,J} \beta_{X,j} \cdot \omega_{j,Y} + \omega_{\varepsilon(X)} \cdot \rho_{\varepsilon(X),Y} \cdot \omega_Y \tag{A-7}$$

We can write $\omega_{j,Y} = \omega_{j,} \cdot \rho_{j,Y} \cdot \omega_Y$, and use the definition $\omega_X \cdot \psi_{X,j} = \beta_{X,j} \cdot \omega_j$ of $\psi_{X,j}$. Then (A-7) becomes:

$$\omega_{X,Y} = \omega_X \cdot \left\{ \sum_{j=1,J} \psi_{X,j} \cdot \rho_{j,Y} \right\} \cdot \omega_Y + \omega_{\varepsilon(X)} \cdot \rho_{\varepsilon(X),Y} \cdot \omega_Y \tag{A-8}$$

In addition, recall from (A-6) that $\omega_{\varepsilon(X)} = \omega_X \cdot \rho_{X,\varepsilon(X)}$, so:

$$\omega_{X,Y} = \omega_X \cdot \left\{ \sum_{j=1,J} \psi_{X,j} \cdot \rho_{j,Y} + \rho_{X,\varepsilon(X)} \cdot \rho_{\varepsilon(X),Y} \right\} \cdot \omega_Y \tag{A-9}$$

Separation of Return Components

Let $\hat{\mathbf{S}}$ be the N by K matrix where each column is a source of risk that we wish to control. Let $\boldsymbol{\chi}$ be the excess return and define portfolio Y with holdings $\mathbf{y}$ determined through $\boldsymbol{\chi} = \boldsymbol{\lambda} \cdot \mathbf{V} \cdot \mathbf{y}$. A regression, as in (A-1):

$$\mathbf{y} = \hat{\mathbf{S}} \cdot \boldsymbol{\beta}_Y + \boldsymbol{\varepsilon}_Y$$

gives us returns we wish to control, $\boldsymbol{\phi} = \lambda \cdot \mathbf{V} \cdot \hat{\mathbf{S}} \cdot \boldsymbol{\beta}_Y$, and the residual we are trying to predict, $\boldsymbol{\theta} = \boldsymbol{\lambda} \cdot \mathbf{V} \cdot \boldsymbol{\varepsilon}_Y$. The portfolios $\mathbf{f} = \hat{\mathbf{S}} \cdot \boldsymbol{\beta}_Y$ and $\mathbf{r} = \boldsymbol{\varepsilon}_Y$ are uncorrelated, and $\boldsymbol{\theta}' \cdot \mathbf{V}^{-1} \cdot \boldsymbol{\phi} = \mathbf{0}$.

New Information Flows in Vintage Example

For each $\tau = 0, T$, calculate $\mathbf{q}(t - \tau)$ by solving, $\boldsymbol{\alpha}(t - \tau) = \lambda \cdot \mathbf{V} \cdot \mathbf{q}(t - \tau)$. Then define:

$$\gamma(t-\tau) = \frac{\mathbf{q}'(t-\tau) \cdot \mathbf{V} \cdot \mathbf{q}(t-\tau-1)}{\mathbf{q}'(t-\tau-1) \cdot \mathbf{V} \cdot \mathbf{q}(t-\tau-1)} \tag{A-10}$$

Our incremental portfolio is:

$$\mathbf{u}(t-\tau) = \mathbf{q}(t-\tau) - \gamma(t-\tau) \cdot \mathbf{q}(t-\tau-1) \tag{A-11}$$

The condition (A-9) guarantees that the new positions are uncorrelated with the preceding portfolio, i.e., $\mathbf{u}'(t - \tau) \cdot \mathbf{V} \cdot \mathbf{q}(t - \tau - 1) = 0$.

Equivalence of Return Regression (49) and Portfolio Regression (54)

If we start with the returns regression:

$$\boldsymbol{\theta} = \mathbf{X} \cdot \mathbf{f} + \mathbf{u}$$

$$\mathbf{f} = \left\{ \mathbf{X}' \cdot \mathbf{V}^{-1} \cdot \mathbf{X} \right\}^{-1} \cdot \mathbf{X}' \cdot \mathbf{V}^{-1} \cdot \boldsymbol{\theta} \tag{A-12}$$

as compared to the portfolio regression:

$$\mathbf{r} = \mathbf{S} \cdot \boldsymbol{\beta}_R + \boldsymbol{\varepsilon}_R$$

$$\boldsymbol{\beta}_R = \left\{ \mathbf{S}' \cdot \mathbf{V} \cdot \mathbf{S} \right\}^{-1} \cdot \mathbf{S}' \cdot \mathbf{V} \cdot \mathbf{r} \tag{A-13}$$

The most direct way is just to substitute $\boldsymbol{\theta} = \lambda \cdot \mathbf{V} \cdot \mathbf{r}$ and $\mathbf{X} = \lambda \cdot \mathbf{V} \cdot \mathbf{S}$ into parts of the formula (A-11) for $\mathbf{f}$. We obtain:

$$\mathbf{X}' \cdot \mathbf{V}^{-1} \cdot \mathbf{X} = \lambda^2 \cdot \mathbf{S}' \cdot \mathbf{V} \cdot \mathbf{S}$$

$$\mathbf{X}' \cdot \mathbf{V}^{-1} \cdot \boldsymbol{\theta} = \lambda^2 \cdot \mathbf{S}' \cdot \mathbf{V} \cdot \mathbf{r} \tag{A-14}$$

Note the risk aversion cancels and plays no role in the calculation, so $\boldsymbol{\beta}_R = \mathbf{f}$.

ENDNOTES

[1]See Grinold and Kahn [2000] or Grinold [2005].

[2]See Clarke, de Silva, and Thorley [2001] and Grinold [2005].

[3]The U.S. dollar is the base currency and is viewed as risk-free.

[4]Results are derived in the appendix.

[5]Not adjusting for degrees of freedom.

[6]The θ_n are the returns we are forecasting. Thus alpha is the forecast or expectation of theta: $\alpha_n = E\langle\theta_n\rangle$. In most cases, the theta is return less market return or could be net of components of return where we are not looking for potential value, added but to control risk. For example, in an equity portfolio we may not want to take any active industry positions. In that context, θ_n would be returns net of industry. See Equation (6).

[7]This can be calculated directly: $OS = \sqrt{\sum_{n=1,N} \sum_{m=1,N} \theta_n \cdot V^{-1}_{n,m} \cdot \theta_m}$ where $V^{-1}_{n,m}$ is the n, m-th element of the inverse of the covariance matrix $\mathbf{V}$. A similar calculation holds for the maximum information ratio: $IR = IR_Q = \sqrt{\sum_{n=1,N} \sum_{m=1,N} \alpha_n \cdot V^{-1}_{n,m} \cdot \alpha_m}$.

[8]If we wish to control industry risk and market risk (i.e., little or no beta exposure), we have to be a bit clever since the market portfolio is a mix of the industry portfolios. Controlling for the market and K-1 of the industries is one approach.

[9]If the asset did not exist or was not in our data set in the prior period, we use zero.

[10]This is to finesse the issue of changes in the covariance matrix from one date to the next. This question can be treated separately by having the sources be portfolio $\mathbf{q}_1$ as produced by a past covariance matrix and portfolio $\mathbf{q}_2$ as produced by the current covariance matrix. Then we could look at sources as the information as captured by $\mathbf{q}_1$ and the change in risk prediction as captured by $\mathbf{q}_2 - \mathbf{q}_1$.

[11]This is a portfolio with holdings h_n with $\sum_{n=1,N} h_n \cdot x_{n,k} = 1$ and $\sum_{n=1,N} h_n \cdot x_{n,j} = 0$ for $j \neq k$ and minimal risk among all portfolios with those attributes.

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